



## APPLICATION OF LAGRANGE THEOREM ON CERTAIN GROUPS

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### ABSTRACT –

*This paper studies about the notion of indecomposable groups. It is proved that if a group is a simple then it is an indecomposable group. If  $G$  is a  $p$ -group of class two, then obtained a necessary and sufficient condition for a group to be indecomposable and also centre of a group is cyclic. If group is non abelian group and the centre of group is not contained in the intersection of maximal sub groups of group then it is a factor direct product group.*

### KEY WORDS:

*cyclic groups, centre of a group, finite Abelian group,  $p$ -group.*

### 1.PRELIMINARIES

**Definition 1.1** [3,4] A group  $A$  is said to be an *ingroup* of a group  $G$  if one of the following holds:

- a)  $A$  is isomorphic to a subgroup of  $G$ .
- b)  $A$  is isomorphic to a factor group of  $G$ .
- c)  $A$  is isomorphic to a factor group of a subgroup of  $G$ .

The notation  $A \leq G^1$  will be used for the above with  $A < G$  denoting that A is an ingroup of G but A is not isomorphic with G. In this case A is said to be a proper ingroup of G.

### **REMARKS[5]**

- . Every homomorphic image of group G is isomorphic to some quotient group of G.
- .If  $f : G \rightarrow G'$  is a homomorphism and onto with kernel then  $G/K \cong G'$  [G' is the homomorphic image of G under f]
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**Definition 1.2:** [3,4] Let  $\{A_\alpha\}$  be a set of groups. A group G is said to be rank 0 if  $G \leq A$  for some  $A \in \{A_\alpha\}$ . G is of rank 1 if G is not of rank 0 and  $G \leq A \times B$  with A and B of rank 0 and  $G \leq A \times B$  with A and B of rank 0. In general G is of rank n. If G has not been assigned rank n-k for  $k > 0$  and  $G \leq C \times D$  with C and D both of ranks less than n. The set of groups with assigned rank is called the closure of  $\{A_\alpha\}$  and is denoted by  $\{\overline{A_\alpha}\}$ .

**Definition 1.3** [3,4] Let G be a group and let  $\{A_\alpha\}$  be the set of all proper ingroups of G. G is called decomposable if  $G \in \{\overline{A_\alpha}\}$ .

**Definition 1.4** [3,4] A group G is called a factordirect product of A and B if for some N,  $G \simeq (A \times B)/N$  and  $1 \neq A, B \subset G$ .

### **P-group**

If a group G has order  $p^m$  where p is a prime number and m is a positive integer, then we say that G is a p-group.

### **SOLABLE GROUP[5]**

A group is said to be Solvable group if  $G^{(k)} = \{e\}$  for some positive integer 'k'.

**Nilpotent group**[5] A group  $G$  is said to be Nilpotent if  $Z_m(G) = G$  for some 'm' the smallest 'm' such that  $Z_m(G) = G$  is called the class of Nilpotency of  $G$ .

**SIMPLE group** [5]

A group  $G$  is called SIMPLE if it has no proper normal sub groups .

**Remarks** [5]

- A group  $G$  is abelian if and only if  $Z(G)=G$ .
- A group of prime order has no proper normal sub groups .
- Every cyclic group is an abelian group.
- Every group of prime order is 'simple'.
- Every group of prime order is cyclic. But converse is not true .
- Prime power group is solvable.
- Every Nilpotent group is solvable.
- $G$  is simple  $\Leftrightarrow G$  has no normal sub groups other than  $G$  and  $\{e\}$ .
- Let  $G$  be a finite group. Then the order of any sub group of  $G$  divides the order of  $G$ .

**Theorem 1**[3,4] Let  $G=H_1/N_1$  with  $H_1$  a sub direct product  $A_1, B_1 \neq 1$ . Then there exist factor groups  $A, B, H$  and  $N$  of  $A_1, B_1, H_1$ , and  $N_1$  respectively such that  $G \cong \frac{H}{N}, H \subset A \times B$  and there exist  $R, S \triangleleft H$  such that  $R \cap S = R \cap N = S \cap N = 1$ . Assume in addition that  $G$  is not a sub direct product and that  $G$  is not isomorphic to a factor group of  $A$  or  $B$ . Then  $G$  contains a normal abelian  $p$ -group.

**Proof** .Consider the element  $S \cap NR$ . Since all of these subgroups are normal  $R \cup N$  may be written as  $RN$  or  $NR$ .

Assume first that  $NR \cap S = 1$ . Since the lattice of normal subgroups of a group is modular it follows that  $N \cup (NR \cap S) = NR \cap NS = N$ . Now if  $NR \neq N$  and  $NS \neq N$  then  $NR \neq NS$  and hence  $H/N$  is a subdirect product. If  $NR = N$  then  $G = \frac{H}{N} \cong \frac{H/R}{N/R}$  and hence  $G$  is

isomorphic to a factor group of A. If  $NS = N$  then  $G = \frac{H}{N} \simeq \frac{H/R}{N/R}$  and G is isomorphic to a factor group of B. Hence If  $NR \cap S = 1$  then G is either a subdirect product or G is isomorphic to a factor group of A. But it is easy to see that  $NR \cap S = 1$  if and only if  $NS \cap R = 1$  which holds if and only if  $RS \cap N = 1$ . For if  $RS \cap N \neq 1$  then there exist r, s and n such that  $rs = n \neq 1$ , and hence  $s = r^{-1}n$  and  $r = ns^{-1}$ . Note that  $r \neq 1$  and  $s \neq 1$  for otherwise  $R \cap N \neq 1$  and  $S \cap N \neq 1$  contrary to Lemma 1.6. Hence if  $Y = NS \cap R$  and  $Z = RS \cap N$  then  $Y \neq 1$  and  $Z \neq 1$ .

The lattice diagram in Figure 1 below illustrates the relationships of the groups that have been discussed above. All of the argument above and some of the argument that follows will verify that the unions and intersections given in the diagram are indeed correct.

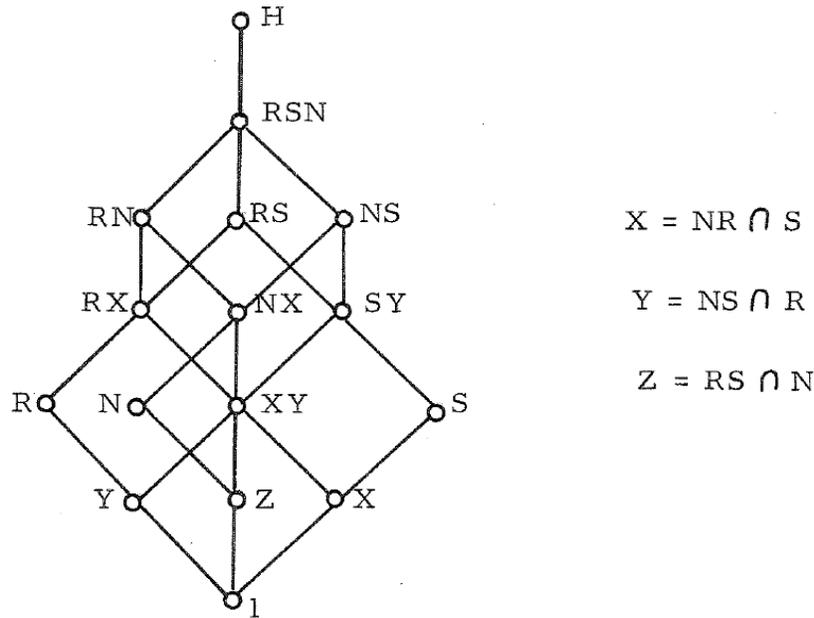


Figure 1

$X \cap Y = (NR \cap S) \cap (NS \cap R) \subseteq R \cap S = 1$ . Applying the same argument to  $X \cap Z$  and  $Y \cap Z$ , (i)  $X \cap Y = X \cap Z = Y \cap Z = 1$ . By the use of the modular law we see:

$XY = (NR \cap S) \cup (NS \cap R) = NS \cap [(NR \cap S) \cup R] = NS \cap NR \cap RS$ . By symmetry the same result holds for  $XZ$  and  $YZ$ . Hence

(ii)  $XY = XZ = YZ = XYZ$

Ore[10] has shown that under conditions (i) and (ii) the group  $XY$  is abelian.

Consider  $XY \cup N = (NS \cap NR \cap RS) \cup N = NS \cap [N \cup (NR \cap S)] = NS \cap [NR \cap (N \cup RS)] = NS \cap [NR \cap NRS] = NS \cap NR$ . Also  $XY \cap N = [NS \cap NR \cap RS] \cap N = N \cap RS = Z$ . Therefore  $\frac{RN \cap SN}{N} = \frac{XY \cap N}{N} \simeq \frac{XY}{XY \cap N} = \frac{XY}{Z} = \frac{XZ}{Z} \simeq X$ . Hence we have exhibited a normal subgroup of  $H/N=G$  isomorphic with  $X$ , which is abelian. Now if  $X$  had composite order then it would contain two characteristic subgroups with trivial intersection. Since a characteristic subgroup of a normal subgroup is normal it would follow that  $G$  is not sub-directly indecomposable. Hence  $X$  is a  $p$ -group. This completes the proof of the theorem.

**Theorem 2.** [3,4] Let  $\{A_\alpha\}$  be a set of groups. If  $G \in \{\overline{A_\alpha}\}$  then  $G/Z(G) \in \{\overline{A_\alpha/Z(A_\alpha)}\}$ .

**Proof** Let  $G$  be of rank 0 in  $\{\overline{A_\alpha}\}$ . This means that  $G \leq A \in \{A_\alpha\}$ . We now consider all the possibilities for  $G \leq A$ .

a) Let  $G \subset A$ . Clearly  $G \cup [Z(A) \cup Z(G)] = G \cup Z(A)$ . Since  $Z(A)Z(G) = Z(G)Z(A)$  the modular law applies and since  $Z(A) \cap G \subseteq Z(G)$ ,  $G \cap [Z(A) \cup Z(G)] = Z(G) \cup [Z(A) \cap G] = Z(G)$ .

Hence  $\frac{G}{Z(G)} \simeq \frac{G}{G} \cap [Z(A) \cup Z(G)] \simeq G \cup [Z(A) \cup Z(G)]/Z(A) \cup Z(G) \simeq G \cup Z(A)/Z(A) \cup Z(G) \simeq G \cup Z(A)/Z(A)/Z(A) \cup Z(G)/Z(A)$ . But  $\cup Z(A)/Z(A) \subset A/Z(A)$ . Hence  $G/Z(G)$  is isomorphic to a factor group of subgroup of  $A/Z(A)$  and  $A \in \{A_\alpha\}$ .

The proceeding argument may be illustrated by the lattice diagram in Figure-2.

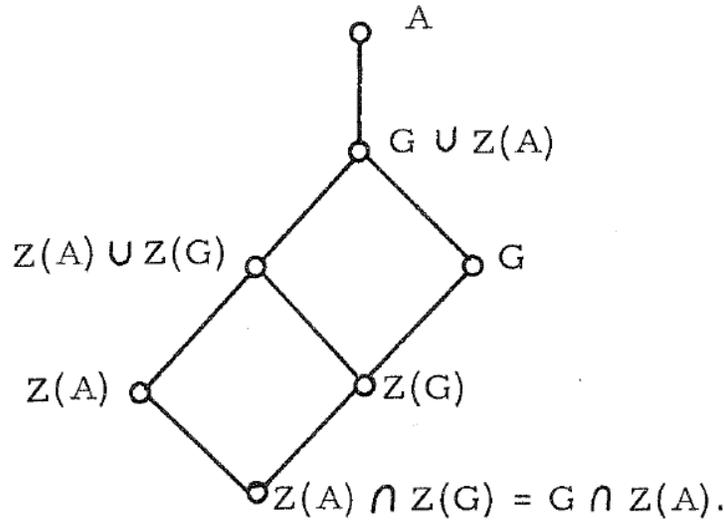


Figure – 2

b) The second case to consider is  $G = A/N$ . Let  $I_a = aZ(A)$  for  $a \in A$  and  $I_{aN} = aNZ(A/N)$ .

We will show that the mapping:

$I_a \rightarrow I_{aN}$  of  $A/Z(A)$  into  $G/Z(G)$  is a homomorphism.

Suppose that  $I_a = 1$  then  $aZ(A) = Z(A)$  and  $a \in Z(A)$ . Consider  $I_{aN} = aNZ(A/N)$ . Since  $a \in Z(A)$ ,  $aN \in Z(A/N)$ . Hence  $I_{aN} = 1$ .

Now let  $I_{a_1} \rightarrow I_{a_1N}$  and

$$I_{a_2} \rightarrow I_{a_2N}$$

$I_{a_1}I_{a_2} = a_1Z(A)a_2Z(A) = a_1a_2Z(A) = I_{a_1a_2}$ , and  $I_{a_1N}I_{a_2N} = I_{a_1a_2N}$ . Hence the mapping is a homomorphism and therefore  $G/Z(G)$  is isomorphic to a factor group of  $A/Z(A)$ .

c) Finally if  $G \simeq H/N$  and  $H \subset A \in \{A_\alpha\}$  then  $H/Z(H)$  is isomorphic to a factor group of a subgroup of  $A/Z(A)$  and  $G/Z(G)$  is isomorphic to a factor group of  $H/Z(H)$ . Hence  $G/Z(G) \in \overline{\{A_\alpha/Z(A_\alpha)\}}$ .

This completes the argument for  $G$  of rank 0. Notice that the above proves that if  $G \leq A$  then  $G/Z(G) \leq A/Z(A)$ .

Now if  $G$  is of rank  $n > 0$  then we assume that for all elements of  $\{\overline{A_\alpha}\}$  of rank less than  $n$  the theorem holds. But  $G \leq C \times D$  with  $C$  and  $D$  of rank less than  $n$ .

Hence  $\frac{C}{Z(C)}, \frac{D}{Z(D)} \in \left\{ \frac{A_\alpha}{Z(A_\alpha)} \right\}$ . But  $\frac{C \times D}{Z(C \times D)} \simeq \frac{C}{Z(C)} \times \frac{D}{Z(D)} \in \left\{ \frac{A_\alpha}{Z(A_\alpha)} \right\}$ . Hence since  $\frac{G}{Z(G)} \leq \frac{C \times D}{Z(C \times D)}, \frac{G}{Z(G)} \in \left\{ \frac{A_\alpha}{Z(A_\alpha)} \right\}$ . The proof is then complete by induction.

**Corollary 1: [3,4]** Let  $G$  be a nilpotent group. If  $G$  is decomposable then  $G$  contains a proper in-group whose class (length of upper central series) is the same as the class of  $G$ .

**Proof** If the class of  $G$  is  $n$ , written  $c(G) = n_1$  and  $Z_r(G)$  is the  $r^{\text{th}}$  element of the upper central series, then  $\frac{G}{Z_n(G)} = 1$  and  $\frac{G}{Z_{n-1}(G)} \supset 1$ . If  $G \in \{\overline{A_\alpha}\}$  and the class of  $A$  is less than  $n$ , then  $\frac{A}{Z_{n-1}(A)} = 1$  for all  $A \in \{A_\alpha\}$ . Therefore  $\left\{ \frac{A_\alpha}{Z_{n-1}(A_\alpha)} \right\} = \{1\}$ . But it follows from the theorem that if  $G \in \{\overline{A_\alpha}\}$  then  $\frac{G}{Z_{n-1}(G)} \in \left\{ \frac{A_\alpha}{Z_{n-1}(A_\alpha)} \right\}$ . Since  $\frac{G}{Z_{n-1}(G)} \supset 1$  this is a contradiction. Hence  $G$  is either not decomposable or it contains an in-group with class  $n$ .

**Corollary 2:[3,4]** Let  $G$  be a soluble group. If  $G$  is decomposable then  $G$  contains a proper in-group whose derived length equals that of  $G$ .

**Proof** Let  $\{A_\alpha\}$  be the set of proper in-groups of  $G$ . If the derived length of  $G$  is  $d$ , then  $G^{(d)} = 1$  but  $G^{(d-1)} \neq 1$ . But it follows from the theorem that  $G^{(d-1)} \in \left\{ A_\alpha^{(d-1)} \right\}$ , and if the derived length of every proper in-group of  $G$  is less than  $d$  then  $G^{(d-1)} \in \left\{ A_\alpha^{(d-1)} \right\} = \{1\}$ , a contradiction. Hence either  $G$  is indecomposable or it contains a proper in-group whose derived length equals that of  $G$ .

**Remarks [3,4,6,7]**

- If  $G$  is a nilpotent group then  $G' \subseteq \emptyset(G)$ .
- If  $G$  is a solvable group then  $G' \subseteq \emptyset(G)$ .
- If  $G$  is nilpotent and  $N \triangleleft G$  then  $N \cap Z(G) \supset 1$ .
- If  $G$  is nilpotent and the class of  $G$  is  $n$  then  $G_{(n)} \subseteq Z(G)$ .
- If  $G$  is solvable and  $N \triangleleft G$  then  $N \cap Z(G) \supset 1$ .

- If  $G$  is solvable and the class of  $G$  is  $n$  then  $G_{(n)} \subseteq Z(G)$ .

**Corollary 1:** If  $G$  is a simple group then  $G$  is indecomposable.

**Proof:** Suppose  $G$  is simple group. If it has no proper sub groups and no proper normal sub groups and no proper in group then  $G$  is not decomposable. therefor  $G$  is indecomposable.

**Corollary2:** Let  $G$  be a  $p$ -group.  $G$  is a sub direct product if and only if  $Z(G)$  is not cyclic.

**Proof:**  $G$  is  $p$ - group. Let us suppose  $G$  is sub direct product .therefore  $G$  has a normal sub groups .By the LAGRANGE THEOREM.(Let  $G$  be a finite group. Then the order of any sub group of  $G$  divides the order of  $G$ .)  $Z(G)$  is also normal sub group of  $G$ . Therefore The order of  $Z(G)$  is also divides group of order  $G$ . The order of  $Z(G)$  is not prime .Therefore  $Z(G)$  is not cyclic .Conversely suppose that  $Z(G)$  is not cyclic. Therefore the order of  $Z(G)$  is not prime.  $Z(G)$  has a proper normal sub groups .Therefore the order of  $Z(G)$  is also divides group of order  $G$ . Therefore the order of  $G$  is not a prime. Therefore  $G$  has a proper normal sub groups. Therefore  $G$  is sub direct product.

**Corollary 3:** If  $G$  is a non-abelian group and  $Z(G) \not\subseteq \emptyset(G)$  then  $G$  is a factor direct product.

**Proof:** Since  $\emptyset(G)$  is the intersection of all maximal subgroups of  $G$ , then there exists one maximal subgroup  $M$  of  $G$  such that  $Z(G) \not\subseteq M$ . Since  $M$  is maximal and  $Z(G)$  is normal in  $G$ , it follows that  $G = MZ(G)$ . But these groups permute element wise and  $Z(G) \subset G$ , hence  $G$  is a factor direct product.

### Note

If  $G$  is a non-abelian group and  $Z(G) \subseteq \emptyset(G)$  then  $G$  is not a factor direct product.

**Corollary 4:** Let  $G$  be a  $p$ -group of class two.  $G$  is indecomposable if and only if  $Z(G)$  is cyclic and  $G$  may be generated by two elements.

**Proof** Suppose  $G$  is indecomposable. Therefore  $G$  is simple.  $G$  is simple  $\Leftrightarrow G$  has no normal sub groups other than  $G$  and  $\{e\}$ . A group of prime order has no proper normal sub groups. Therefore  $G$  has a prime order group. Every group of prime order is cyclic. Therefore  $G$  is cyclic.  $Z(G)$  is sub group of  $G$  and  $Z(G)=G$ . Since  $G$  is cyclic.  $Z(G)$  is cyclic. Conversely suppose that  $Z(G)$  is cyclic and  $Z(G)=G$ . Since given  $G$  is a  $p$ -group and  $G$  is cyclic.  $G$  has no proper normal sub groups. Therefore  $G$  has prime order. Therefore  $G$  is simple. Therefore  $G$  is indecomposable.

### CONCLUSION

It follows from Theorem 1. that a decomposable groups is either a sub direct product or it contains a normal abelian  $p$ -groups. At this stage, however, it is not clear how useful it is to know that groups with no normal abelian  $p$ -subgroup are either decomposable or sub direct product. Certainly this question might be profitably pursued. One might begin with composite order groups which have abelian Sylow subgroups.

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